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Approximation Formulae of Voronovskaya-Type for Certain Convolution Operators

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1. INTRODUCTION

In this paper we consider approximation properties of operators U_ρ of convolution type of which the kernel is the ρ-th power of a function $\beta(t)$ belonging to a class B. The operators U_ρ are acting on the elements $f(t)$ of a class M of functions and are defined by

$$U_\rho(f; x) = \frac{1}{I_\rho} \int_{-\infty}^{\infty} f(x-t)\, \beta^\rho(t)\, dt. \tag{1}$$

The class B consists of all real functions $\beta(t)$ defined on the whole real line $\mathbf{R}$ and possessing the following four properties 1.–4.:

1. $\beta(t) \geqslant 0$ on $\mathbf{R}$.
2. $\beta(t)$ is continuous at $t = 0$, $\beta(0) = 1$.
3. For each $\delta > 0$, $\sup_{|t| \geqslant \delta} \beta(t) < 1$.
4. $\beta(t)$ belongs to the Lebesgue class L_1, i.e., $\int_{-\infty}^{\infty} \beta(t)\, dt$ exists in the sense of Lebesgue.

We set

$$I_\rho = \int_{-\infty}^{\infty} \beta^\rho(t)\, dt \; (\rho \geqslant 1). \tag{2}$$

The class M consists of all real functions $f(t)$, defined, bounded and Lebesgue-measurable on $\mathbf{R}$. Then the right-hand side of (1) exists for all $\rho \geqslant 1$. Clearly, the operators U_ρ are linear and positive on M.

Two main questions will be answered in this paper. Firstly, for the operators U_ρ with $\beta(t) \in B$, $f(t) \in M$ and continuous at $t = x$, it is proved in Theorem 1 that if $\rho \to \infty$,

$$U_\rho(f; x) - f(x) \to 0. \tag{3}$$

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Secondly, for the speed with which $U_\rho(f; x) - f(x)$ tends to zero if $\rho \to \infty$, asymptotic formulae of Voronovskaya type are derived under conditions which imply that more is known about the behaviour of $\beta(t)$ for $t \downarrow 0$ and $t \uparrow 0$, respectively (property 5., resp. 5'.) and that $f''(x)$ exists. It turns out, that, in some situations, with respect to this behaviour of $\beta(t)$, in the asymptotic formulae only $f'(x)$ comes up, in other ones both $f'(x)$ and $f''(x)$ and in still others only $f''(x)$. Theorems 2–8 are devoted to this study, Theorem 7 being of special interest.

In some very special cases of operators of the type U_ρ Voronovskaya type formulae were already known. To the best of our knowledge in all of them $\beta(t)$ is continuous and even. Some examples of such operators are considered in the last section in the context of our general results. Also a number of more general operators are treated there.

From the point of view of approximation theory Korovkin [4] was the first to study a special case of operators of the type U_ρ. However, in his study the interval of integration is finite and $\beta(t)$ is everywhere continuous. For a bounded $f(t)$, which is continuous at $t = x$, he proved (3) for $\rho \in \mathbf{N}$. In the literature some particular operators of type (1) occur much earlier: e.g. Weierstrass [10] used such operators with $\beta(t) = e^{-t^2}$ and $\rho \in \mathbf{N}$ to prove his celebrated approximation theorem, while Landau [5] proved the same theorem, using $\beta(t) = 1 - t^2$ $(|t| \leqslant 1)$, $\beta(t) = 0$ $(|t| > 1)$, $\rho \in \mathbf{N}$. Of other authors who incidentally used special operators of the above form we only mention here Titchmarsh [8] and Bochner [1]. In their 1970 book [3] Butzer and Nessel consider in chapter 3 a.o. some particular cases of the operators (1). For them they prove (3) if $f \in L^\infty$, f continuous at $t = x$. In order to investigate the speed of convergence in (3) (in the sense of the present paper), in case the right-hand side of (1) is of Fejér's type they assume that $\beta(t)$ is even.

In 1973 Bojanic and Shisha [2] continuing the work on the special type of operators U_ρ studied by Korovkin, used a special form of property 5. below in deriving a formula for the speed with which $U_\rho(f; x) - f(x)$ tends to zero if $\rho \to \infty$ $(\rho \in \mathbf{N})$. They assumed $\beta(t)$ to be even, continuous and monotonically decreasing for $t \geqslant 0$ (they consider only a finite interval of integration). The direction of their work is different from ours. They assumed $f(t)$ to be continuous and they made use of the modulus of continuity of f.

2. Some Lemmas

In Lemmas 1–5 it is assumed that $\beta(t) \in \boldsymbol{B}$ and $\nu = 0, 1, 2, \ldots$. We put for $\delta > 0$ and $\rho \geqslant 1$

$$I_{\nu\rho}(\delta) = \int_{-\delta}^{\delta} t^\nu \beta^\rho(t)\, dt, \quad A_{\nu\rho}(\delta) = \int_0^{\delta} t^\nu \beta^\rho(t)\, dt, \quad R_\rho(\delta) = \int_{|t| \geqslant \delta} \beta^\rho(t)\, dt. \tag{4}$$

In case $\nu = 0$, we shall write

$$I_\rho(\delta) = I_{0\rho}(\delta). \tag{5}$$

LEMMA 1. *If $\delta > 0$, $\delta \geqslant \eta > 0$ and if ν odd, $\beta(t)$ not even on an arbitrary small interval around $t = 0$, then*

$$\lim_{\rho\to\infty} \frac{I_{\nu\rho}(\delta)}{I_{\nu\rho}(\eta)} = 1. \tag{6}$$

Proof. If $\delta = \eta$ (6) is trivial. If $\delta \neq \eta$, it may be supposed that $\delta > \eta$. Then

$$I_{\nu\rho}(\delta) = I_{\nu\rho}(\eta) + \int_{\eta\leqslant|t|\leqslant\delta} t^\nu \beta^\rho(t)\,dt, \tag{7}$$

where if p is even

$$0 \leqslant \int_{\eta\leqslant|t|\leqslant\delta} t^\nu\beta^\rho(t)\,dt \leqslant 2\delta^{\nu+1}\{\sup_{\eta\leqslant|t|\leqslant\delta} \beta(t)\}^\rho = 2\delta^{\nu+1}(1-\tau)^\rho, \tag{8}$$

τ satisfying the inequality $0 < \tau < 1$, because of property 3. By property 2. there exists a positive number ξ ($\xi \leqslant \eta$) such that $\beta(t) \geqslant 1 - \frac{1}{2}\tau$ for all t with $|t| \leqslant \xi$. Consequently

$$I_{\nu\rho}(\eta) \geqslant I_{\nu\rho}(\xi) \geqslant 2(1-\tfrac{1}{2}\tau)^\rho \int_0^\xi t^\nu\,dt = \frac{2\xi^{\nu+1}}{\nu+1}(1-\tfrac{1}{2}\tau)^\rho. \tag{9}$$

From (7), (8) and (9) then follows that

$$0 \leqslant \frac{I_{\nu\rho}(\delta)}{I_{\nu\rho}(\eta)} - 1 \leqslant (\nu+1)\left(\frac{1-\tau}{1-\frac{1}{2}\tau}\right)^\rho (\delta/\xi)^{\nu+1}.$$

This proves Lemma 1 if p is even. If p is odd a similar reasoning holds.

LEMMA 2. *If $\delta > 0$ then*

$$\lim_{\rho\to\infty} \frac{R_\rho(\delta)}{I_\rho(\delta)} = 0. \tag{10}$$

Proof. According to property 3. there exists a number τ with $0 < \tau < 1$ such that $0 \leqslant \beta(t) \leqslant 1 - \tau$ for all t with $|t| \geqslant \delta$. Then, if $\rho \geqslant 2$,

$$R_{\nu\rho}(\delta) \leqslant (1-\tau)^{\rho-1} \int_{|t|\geqslant\delta} \beta(t)\,dt \leqslant (1-\tau)^{\rho-1}\,\|\beta\|, \tag{11}$$

where $\|\beta\|$ is the L_1-norm of $\beta(t)$, which exists because of property 4. On account of property 2. there exists a positive number η, such that $\beta(t) \geqslant 1 - \frac{1}{2}\tau$ for all t with $|t| \leqslant \eta$ and hence

$$I_\rho \geqslant I_\rho(\eta) \geqslant 2\eta(1 - \tfrac{1}{2}\tau)^\rho. \tag{12}$$

Because of (11) and (12) it follows that

$$0 \leqslant \frac{R_\rho(\delta)}{I_\rho} \leqslant \frac{\|\beta\|}{2\eta(1-\tau)} \left(\frac{1-\tau}{1-\frac{1}{2}\tau}\right)^\rho$$

and thus (10).

LEMMA 3. *If* $\delta > 0$, *then*

$$\lim_{\rho\to\infty} \frac{I_\rho(\delta)}{I_\rho} = 1 \text{ and } \lim_{\rho\to\infty} \frac{R_\rho(\delta)}{I_\rho} = 0. \tag{13}$$

Proof. From

$$I_\rho(\delta) + R_\rho(\delta) = I_\rho\,,$$

$$0 \leqslant \frac{R_\rho(\delta)}{I_\rho} \leqslant \frac{R_\rho(\delta)}{I_\rho(\delta)}$$

and Lemma 2, (13) follows.

LEMMA 4.

$$\lim_{\rho\to\infty} \frac{I_{\rho+1}}{I_\rho} = 1. \tag{14}$$

Proof. Because of Property 2. there exists to every $\epsilon > 0$ a $\delta > 0$ such, that for all t with $|t| \leqslant \delta$ the relation $0 \leqslant 1 - \beta(t) < \epsilon/2$ holds. Hence

$$\begin{aligned} I_{\rho+1} \leqslant I_\rho &= \int_{-\infty}^{\infty} (1 - \beta(t))\, \beta^\rho(t)\, dt + I_{\rho+1} \\ &= \int_{-\delta}^{\delta} (1 - \beta(t))\, \beta^\rho(t)\, dt + R_\rho(\delta) - R_{\rho+1}(\delta) + I_{\rho+1} \\ &< (\epsilon/2)\, I_\rho + R_\rho(\delta) + I_{\rho+1}\,. \end{aligned}$$

By Lemma 3 this means that for all sufficiently large ρ

$$0 \leqslant 1 - \frac{I_{\rho+1}}{I_\rho} < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, (14) follows.

LEMMA 5. *If* $\delta > 0$ *and* $\eta > 0$, *then*

$$\lim_{\rho\to\infty} \frac{A_{\nu\rho}(\delta)}{A_{\nu\rho}(\eta)} = 1.$$

Proof. It can be given similarly to that of Lemma 1.

A lemma of a different character, which is useful in the next sections is the following one.

LEMMA 6. *If* $\delta > 0$, $\lambda \geqslant 0$, $\sigma > 0$, $\alpha > 0$, *then*

$$\lim_{\rho\to\infty} \rho^{(\lambda+1)/\alpha} \int_0^\delta t^\lambda e^{-\rho\sigma t^\alpha}\, dt = \alpha^{-1}\sigma^{-(\lambda+1)/\alpha}\Gamma((\lambda+1)/\alpha). \tag{15}$$

Proof. (15) readily follows by substituting $\rho\sigma t^\alpha = u$ in the integral.

3. THE APPROXIMATION THEOREM

In this section we prove the following theorem:

THEOREM 1. *If* $\beta(t) \in B$, $f(t) \in M$ *and* $f(t)$ *is continuous at a point* $t = x$, *then*

$$\lim_{\rho\to\infty} U_\rho(f; x) = f(x).$$

Proof. Since $f(t)$ is continuous at $t = x$, there exists to every $\epsilon > 0$ a $\delta > 0$ such that for all t with $|t| \leqslant \delta$

$$|f(x-t) - f(x)| < \epsilon/2.$$

Because of property 3. there exists a constant $M > 0$, such that for all t with $|t| \geqslant \delta$

$$|f(x-t) - f(x)| < M(1 - \beta(t)).$$

Consequently, for all t

$$|f(x-t) - f(x)| < (\epsilon/2) + M(1 - \beta(t)).$$

Applying the operator U_ρ it follows from its linearity and positivity that

$$|U_\rho(f; x) - f(x)| < (\epsilon/2) + M\left(1 - \frac{I_{\rho+1}}{I_\rho}\right).$$

By Lemma 4 this means that for all sufficiently large ρ

$$|U_\rho(f; x) - f(x)| < \epsilon$$

which proves the theorem.

4. The Speed of Approximation

In determining an asymptotic expression for the speed with which the image $U_\rho(f; x)$ tends to $f(x)$ if $\rho \to \infty$, at a point $t = x$ of continuity of $f(t)$, we assume that $f''(x)$ exists.

Because of the existence of $f''(x)$ we can write

$$f(x - t) - f(x) = -tf'(x) + \tfrac{1}{2}t^2 f''(x) + t^2 \gamma_x(t), \tag{16}$$

where $\gamma_x(t)$ is bounded on $\mathbf{R}$ and with the definition $\gamma_x(0) = 0$, $\gamma_x(t)$ is continuous at $t = 0$. Consequently, to each $\eta > 0$ there exists a $\delta > 0$, such that for all t with $|t| \leqslant \delta$ the inequality

$$|\gamma_x(t)| < \eta \tag{17}$$

holds. Then, with the notation (4),

$$\begin{aligned} U_\rho(f; x) - f(x) &= \frac{1}{I_\rho} \int_{-\infty}^{\infty} \{f(x - t) - f(x)\}\, \beta^\rho(t)\, dt \\ &= \frac{1}{I_\rho} \Big[\int_{-\delta}^{\delta} \{-tf'(x) + \tfrac{1}{2}t^2 f''(x) + t^2\gamma_x(t)\}\, \beta^\rho(t)\, dt \\ &\quad + \int_{|t|\geqslant\delta} \{f(x - t) - f(x)\}\, \beta^\rho(t)\, dt \Big] \\ &= -f'(x)\frac{I_{1\rho}(\delta)}{I_\rho} + \tfrac{1}{2}f''(x)\frac{I_{2\rho}(\delta)}{I_\rho} + \frac{J_\rho(\delta)}{I_\rho} + \frac{K_\rho(\delta)}{I_\rho}, \end{aligned} \tag{18}$$

where

$$J_\rho(\delta) = \int_{-\delta}^{\delta} t^2 \gamma_x(t)\, \beta^\rho(t)\, dt \tag{19}$$

and

$$K_\rho(\delta) = \int_{|t|\geqslant\delta} \{f(x - t) - f(x)\}\, \beta^\rho(t)\, dt. \tag{20}$$

In what follows the asymptotic behaviour for $\rho \to \infty$ of $I_{1\rho}(\delta)/I_\rho$, $I_{2\rho}(\delta)/I_\rho$, $J_\rho(\delta)/I_\rho$ and $K_\rho(\delta)/I_\rho$ respectively, will be determined. The results, giving the asymptotic behaviour of (18) for $\rho \to \infty$, will be given in Theorems 2–8.

In addition to properties 1.–4. it is now assumed that $\beta(t)$ possesses the following property 5. which makes the behaviour of $\beta(t)$ for $t \to 0$ more precise and which allows this behaviour to be different if t tends from the positive or from the negative side to $t = 0$:

5. $\beta(t) = 1 - ct^{\alpha} + \phi(t)$ it $t \downarrow 0$, with $\alpha > 0$, $c > 0$, $\phi(t) = o(t^{\alpha})$;

$$\beta(t) = 1 - c' \mid t \mid^{\alpha'} + \psi(t) \text{ it } t \uparrow 0, \text{ with } \alpha' > 0, c > 0, \psi(t) = o(\mid t \mid^{\alpha'}). \tag{21}$$

Obviously, it will be necessary to investigate the three cases $\alpha > \alpha'$, $\alpha < \alpha'$ and $\alpha = \alpha'$ separately, while in the latter case distinction has to be made between $c \neq c'$ and $c = c'$. In the following parts of Section 4 the cases $\alpha > \alpha'$, $\alpha < \alpha'$ and $\alpha = \alpha'$ with $c \neq c'$ will be treated. Sub-section 4.2 is devoted to a common treatment of these three cases as far as possible; in Sub-section 4.3 theorems will be derived from the results of Sub-section 4.2 for each of the cases $\alpha > \alpha'$, $\alpha < \alpha'$ and $\alpha = \alpha'$ with $c \neq c'$, separately. The case $\alpha = \alpha'$, $c = c'$ is investigated in Sub-section 4.4.

4.2. Asymptotic Behaviour of (18)

Although in studying the asymptotic behaviour of $I_{\nu\rho}(\delta)$ $(\delta > 0)$ if $\rho \to \infty$, only the cases $\nu = 0$, 1 and 2 are of direct interest, it will be assumed, that ν is a non-negative integer. Then

$$I_{\nu\rho}(\delta) = \int_{-\delta}^{\delta} t^{\nu}\beta^{\rho}(t)\,dt = \int_{0}^{\delta} t^{\nu}\beta^{\rho}(t)\,dt + (-1)^{\nu}\int_{0}^{\delta} t^{\nu}\beta^{\rho}(-t)\,dt, \quad (\rho \geqslant 1). \tag{22}$$

Because of property 2. there exists a constant δ_0 with $0 < \delta_0 \leqslant \delta$ such that on the interval $0 \leqslant t \leqslant \delta_0$ both $\beta(t) > 0$ and $\beta(-t) > 0$. Then, by (22) with δ replaced by δ_0,

$$\begin{aligned} I_{\nu\rho}(\delta_0) &= \int_{0}^{\delta_0} t^{\nu}e^{\rho\log\beta(t)}\,dt + (-1)^{\nu}\int_{0}^{\delta_0} t^{\nu}e^{\rho\log\beta(-t)}\,dt \\ &= A_{\nu\rho}(\delta_0) + (-1)^{\nu}B_{\nu\rho}(\delta_0). \end{aligned} \tag{23}$$

Again, on account of property 5. there exists to each ϵ with

$$0 < \epsilon < \min(c, c')$$

a δ_ϵ with $0 < \delta_\epsilon \leqslant \delta_0$ such that for all t satisfying $0 \leqslant t \leqslant \delta_\epsilon$ both relations

$$\begin{aligned} -(c + \epsilon)\,t^{\alpha} &\leqslant \log\beta(t) \leqslant -(c - \epsilon)\,t^{\alpha}, \\ -(c' + \epsilon)\,t^{\alpha} &\leqslant \log\beta(-t) \leqslant -(c' - \epsilon)\,t^{\alpha} \end{aligned}$$

hold. Consequently, $A_{\nu\rho}(\delta_\epsilon)$ satisfies the inequalities

$$\int_0^{\delta_\epsilon} t^\nu e^{-\rho(c+\epsilon)t^\alpha}\,dt \leqslant A_{\nu\rho}(\delta_\epsilon) \leqslant \int_0^{\delta_\epsilon} t^\nu e^{-\rho(c-\epsilon)t^\alpha}\,dt.$$

Applying Lemma 6 it follows with the notation

$$\alpha^{-1}(\nu+1) = a \tag{24}$$

that

$$\alpha^{-1}(c+\epsilon)^{-a}\,\Gamma(a) \leqslant \liminf_{\rho\to\infty} \rho^a A_{\nu\rho}(\delta_\epsilon) \leqslant \limsup_{\rho\to\infty} \rho^a A_{\nu\rho}(\delta_\epsilon) \leqslant \alpha^{-1}(c-\epsilon)^{-a}\,\Gamma(a). \tag{25}$$

By writing

$$A_{\nu\rho}(\delta_{\epsilon_1}) = A_{\nu\rho}(\delta_\epsilon) - \{A_{\nu\rho}(\delta_\epsilon) - A_{\nu\rho}(\delta_{\epsilon_1})\}\ (0 < \delta_{\epsilon_1} < \delta_\epsilon)$$

it is, on account of property 3., clear, that $\lim_{\rho\to\infty} \rho^a\{A_{\nu\rho}(\delta_\epsilon) - A_{\nu\rho}(\delta_{\epsilon_1})\} = 0$ and this means that both the lim inf and the lim sup in (25) are independent of $\delta_\epsilon(0 < \delta_\epsilon \leqslant \delta_0)$. If then, ϵ runs through a monotonically decreasing null-sequence and the sequence of corresponding δ_ϵ is chosen to be also a monotonically decreasing null-sequence, it follows that

$$\lim_{\rho\to\infty} \rho^a A_{\nu\rho}(\delta_0) = \alpha^{-1}c^{-a}\Gamma(a). \tag{26}$$

Similarly,

$$\lim_{\rho\to\infty} \rho^{a'} B_{\nu\rho}(\delta_0) = (\alpha')^{-1}\,(c')^{-a'}\,\Gamma(a'), \tag{27}$$

where

$$(\alpha')^{-1}(\nu+1) = a'. \tag{28}$$

Combining (23), (26), (27) and applying Lemma 5 to (26) and (27) the following result is arrived at

$$I_{\nu\rho}(\delta) = \alpha^{-1}c^{-a}\Gamma(a)\,\rho^{-a} + (-1)^\nu\,(\alpha')^{-1}\,(c')^{-a'}\,\rho^{-a'} + o(\rho^{-a}) + o(\rho^{-a'}) \tag{29}$$

where $\nu = 0, 1, 2,\ldots$, and a, a' are given by (24), (28) respectively.

Considering $J_\rho(\delta)$ and $K_\rho(\delta)$ defined in (19) and (20), it follows from (17) that

$$|\,J_\rho(\delta)| \leqslant \eta I_{2\rho}(\delta) \tag{30}$$

and since $f \in M$ there exists a constant $P > 0$, such that for all t on R $|f(t)| \leqslant \frac{1}{2}P$ and hence, by (4) and (11)

$$|\,K_\rho(\delta)| \leqslant PR_\rho(\delta) \leqslant P\,\|\,\beta\,\|(1-\tau)^{\rho-1} \qquad (0 < \tau < 1). \tag{31}$$

4.3. Asymptotic Behaviour of (18) if not $\alpha = \alpha', c = c'$

THEOREM 2. *If $\beta(t) \in B$ and $\beta(t)$ possesses property 5. with $\alpha > \alpha'$, if $f(t) \in M$ and if $f''(x)$ exists at a point $t = x$, then*

$$\rho^{1/\alpha}\{U_\rho(f; x) - f(x)\} = -c^{-1/\alpha}\Gamma(2/\alpha)\{\Gamma(1/\alpha)\}^{-1}f'(x) + o(1) \quad (\rho \to \infty). \tag{32}$$

Proof. As $\alpha > \alpha'$, it follows from (24) and (28) that $a < a'$. This means that for $\nu = 0, 1, \ldots$, (29) becomes

$$I_{\nu\rho}(\delta) = \alpha^{-1}c^{-a}\Gamma(a)\,\rho^{-a} + o(\rho^{-a}) \qquad (\rho \to \infty). \tag{33}$$

Hence, using (5), (24) and (13),

$$\frac{I_{1\rho}(\delta)}{I_\rho} = c^{-1/\alpha}\Gamma(2/\alpha)\{\Gamma(1/\alpha)\}^{-1}\,\rho^{-1/\alpha} + o(\rho^{-1/\alpha})\;(\rho \to \infty), \tag{34}$$

$$\frac{I_{2\rho}(\delta)}{I_\rho} = c^{-2/\alpha}\Gamma(3/\alpha)\{\Gamma(1/\alpha)\}^{-1}\,\rho^{-2/\alpha} + o(\rho^{-2/\alpha})\;(\rho \to \infty), \tag{35}$$

and, on account of (30), (35), (31), (33) with $\nu = 0$,

$$\frac{J_\rho(\delta)}{I_\rho} = \mathcal{O}(\rho^{-2/\alpha}), \frac{K_\rho(\delta)}{I_\rho} = \mathcal{O}(\rho^{1/\alpha})(1 - \tau)^{\rho-1})\;(\rho \to \infty). \tag{36}$$

Substituting (34), (35) and (36) in (18), (32) follows.

In case $\alpha' > \alpha$ the following theorem holds:

THEOREM 3. *If $\beta(t) \in B$ and $\beta(t)$ possesses property 5. with $\alpha' > \alpha$, if $f(t) \in M$ and if $f''(x)$ exists at a point $t = x$, then*

$$\rho^{1/\alpha}\{U_\rho(f; x) - f(x)\} = c^{-1/\alpha'}\Gamma(2/\alpha')\{\Gamma(1/\alpha')\}^{-1}f'(x) + o(1) \qquad (\rho \to \infty). \tag{37}$$

Proof. As $\alpha' > \alpha$, it follows from (24) and (28) that $a' < a$ and hence (29) becomes

$$I_{\nu\rho}(\delta) = (-1)^\nu(\alpha')^{-1}(c')^{-a'}\Gamma(a')\,\rho^{-a'} + o(\rho^{-a'}) \qquad (\rho \to \infty).$$

Then the proof of (37) can be continued in an analogous way as that of (32) from (33) onwards.

In case $\alpha = \alpha'$, $c \neq c'$ we have

THEOREM 4. *If $\beta(t) \in B$ and $\beta(t)$ possesses property 5. with $\alpha = \alpha'$, $c \neq c'$, if $f(t) \in M$ and if $f''(x)$ exists at a point $t = x$, then*

$$\rho^{1/\alpha}\{U_\rho(f; x) - f(x)\}$$
$$= \Gamma(2/\alpha)\{\Gamma(1/\alpha)\}^{-1}(cc')^{-1/\alpha}\{c^{1/\alpha} - (c')^{1/\alpha}\}f'(x) + o(1) \qquad (\rho \to \infty). \tag{38}$$

Proof. From (24) and (28) it follows that $a = a'$. Therefore (29) becomes for $\nu = 0, 1, \ldots,$

$$I_{\nu\rho}(\delta) = \alpha^{-1}\Gamma(a)\{c^{-a} + (-1)^{\nu}(c')^{-a}\}\,\rho^{-a} + o(\rho^{-a}) \qquad (\rho \to \infty) \tag{39}$$

and the proof of (38) can be continued in an analogous way as that of (32) from (33) onwards.

4.4. Asymptotic Behaviour of (18) if $\alpha = \alpha', c = c'$

THEOREM 5. *If $\beta(t) \in B$ and $\beta(t)$ possesses property* 5. *with $\alpha = \alpha'$, $c = c'$, if $f(t) \in M$ and if $f''(x)$ exists at a point $t = x$, then*

$$\rho^{1/\alpha}\{U_\rho(f;\, x) - f(x)\} = o(1) \qquad (\rho \to \infty). \tag{40}$$

Proof. From (24) and (28) it follows that $a = a'$ and because $c = c'$, (29) gives for ν odd

$$I_{\nu\rho}(\delta) = o(\rho^{-a}) \qquad (\rho \to \infty) \tag{41}$$

and the proof of (40) can be continued in an analogous way as that of (32) from (33) onwards.

It should be noticed that a special case of that with which Theorem 5 deals is that case where in property 5. not only $\alpha = \alpha'$, $c = c'$, but also $\phi(t) = \psi(-t)$ on an interval $0 \leqslant t < \xi$. Then $\beta(t)$ is an even function on $|\,t\,| \leqslant \theta$, where $\theta = \min(\xi, \delta)$. By (22) $I_{1\rho}(\theta) = 0$, while (35), (30), (36) still hold. Using these results, (18) gives

$$|\,\rho^{2/\alpha}\{U_\rho(f;\, x) - f(x)\} - 2^{-1}Qf''(x)| < \eta Q + o(1) \qquad (\rho \to \infty),$$

where η is used in (17) and

$$Q = c^{-2/\alpha}\Gamma(3/\alpha)\{\Gamma(1/\alpha)\}^{-1}.$$

This leads to

THEOREM 6. *If $\beta(t) \in B$ and $\beta(t)$ is even in a neighborhood of $t = 0$, if $f(t) \in M$ and if $f''(x)$ exists at a point $t = x$, then*

$$\rho^{2/\alpha}\{U_\rho(f;\, x) - f(x)\} = 2^{-1}c^{-2/\alpha}\Gamma(3/\alpha)\{\Gamma(1/\alpha)\}^{-1}f''(x) + o(1) \qquad (\rho \to \infty). \tag{42}$$

Remark. In many examples of known operators of convolution type $\beta(t)$ is even on the whole of the real axis. Then Theorem 6 holds a fortiori. Viz. Section 6.

From the above it is clear that if a more precise result is desired then Theorem 5 gives, it will be necessary that in Property 5 more is known about $\phi(t)$ and $\psi(t)$ in a neighborhood of $t = 0$. Theorem 6 is already an example of this.

In what follows an investigation of $I_{\nu\rho}(\delta)$, as defined in (22), will be given under the condition that $\beta(t)$ possesses Property 5 with $\alpha = \alpha'$, $c = c'$, $\phi(t) \neq \psi(-t)$ on an interval $0 < t < \delta$ $(\delta > 0)$. Of course this investigation will again lead to Theorem 5, but if more is known about the way in which $\phi(t)$ and $\psi(-t)$ tend to zero if $t \downarrow 0$, it leads to results which are more precise than Theorem 5. An important example is studied in Section 5.

Let in Property 5

$$X(t) = \psi(-t) - \phi(t) \tag{43}$$

be either positive or negative for all sufficiently small values of $t > 0$. Thus, let Δ $(0 < \Delta \leqslant \delta)$ be chosen so small that either

$$X(t) > 0 \qquad (\text{for all } t \text{ with } 0 < t \leqslant \Delta)$$

or

$$X(t) < 0 \qquad (\text{for all } t \text{ with } 0 < t \leqslant \Delta),$$

and, moreover,

$$1 - ct^{\alpha} + \phi(t) > \tfrac{1}{2} \quad \text{and} \quad 1 - ct^{\alpha} + \psi(-t) > \tfrac{1}{2} \quad (0 \leqslant t \leqslant \Delta). \tag{44}$$

Let then

$$\tau = sgn\, X(t) \qquad (0 < t \leqslant \Delta). \tag{45}$$

On this interval τ is constant.

In case ν even, it follows from (39) with $c = c'$, that

$$I_{\nu\rho}(\Delta) = 2\alpha^{-1}c^{-a}\Gamma(a)\,\rho^{-a} + o(\rho^{-a}), \qquad a = \alpha^{-1}(\nu + 1), \tag{46}$$

from which, with $\nu = 2$ and $\nu = 0$, (5) and Lemma 3,

$$\frac{I_{2\rho}(\Delta)}{I_{\rho}} = c^{-2/\alpha}\Gamma(3/\alpha)\{\Gamma(1/\alpha)\}^{-1}\,\rho^{-2/\alpha} + o(\rho^{-2/\alpha})\ (\rho \to \infty). \tag{47}$$

Next the case ν is odd is studied. Then, $I_{\nu\rho}(\Delta)$ is written as

$$I_{\nu\rho}(\Delta) = \int_0^{\Delta} t^{\nu}\,\{e^{\rho\log(1-ct^{\alpha}+\phi(t))} - e^{\rho\log(1-ct^{\alpha}+\psi(-t))}\}\,dt$$

$$= \int_0^{\Delta} t^{\nu}e^{\rho\log(1-ct^{\alpha}+\phi(t))}\,\{1 - e^{\rho\log(1+X(t)/(1-ct^{\alpha}+\phi(t)))}\}\,dt. \tag{48}$$

Writing for $0 \leqslant t \leqslant \Delta$

$$\log(1 - ct^\alpha + \phi(t)) = -ct^\alpha + \xi(t), \tag{49}$$

$$\log\left(1 + \frac{X(t)}{1 - ct^\alpha + \phi(t)}\right) = X(t) + \eta(t), \tag{50}$$

then

$$\xi(t) = o(t^\alpha), \eta(t) = \mathcal{O}(t^\alpha X(t)) \qquad (t \downarrow 0). \tag{51}$$

Since $\beta(t)$ is bounded on $\mathbf{R}$, $\phi(t)$ and $\psi(-t)$ are bounded on the interval $0 \leqslant t \leqslant \Delta$ and by (43) $X(t)$ too. Then it follows from (51) that there exists on this interval a bounded, monotonically increasing function $\zeta(t)$ with $\zeta(0) = 0$, and a positive constant r such that for all t of this interval

$$|\xi(t)| \leqslant t^\alpha \zeta(t), \tag{52}$$

$$|\eta(t)| \leqslant rt^\alpha |X(t)|. \tag{53}$$

Assertion. It is possible to construct for all sufficiently large values of ρ, say $\rho \geqslant Q$, two positive functions $\epsilon(\rho)$ and $\delta(\rho)$, both monotonically decreasing to zero if $\rho \to \infty$, with $\epsilon(Q) < \min(c, 1)$ and $\delta(Q) < \Delta$ such that for all $\rho \geqslant Q$ and all t with $0 \leqslant t \leqslant \delta(\rho)$ both

$$-(c + \epsilon(\rho))\, t^\alpha \leqslant -ct^\alpha + \xi(t) \leqslant -(c - \epsilon(\rho))\, t^\alpha \tag{54}$$

and

$$-\rho(1 + \tau\epsilon(\rho))X(t) \leqslant 1 - e^{\rho(X(t)+\eta(t))} \leqslant -\rho(1 - \tau\epsilon(\rho))X(t) \tag{55}$$

hold, with τ given in (45).

Obviously, (54) is satisfied if

$$\zeta(\delta(\rho)) \leqslant \epsilon(\rho) \qquad (\rho \geqslant Q). \tag{56}$$

Investigating (55) we write for $0 \leqslant t \leqslant \Delta$

$$1 - e^{\rho(X(t)+\eta(t))} = -\rho(X(t) + \eta(t)) + \rho X(t) y(t, \rho) \tag{57}$$

which transforms (55) into

$$|-\rho\eta(t) + \rho X(t) y(t, \rho)| \leqslant \rho\epsilon(\rho)\, \tau X(t) \qquad (0 \leqslant t \leqslant \delta(\rho)). \tag{58}$$

Because of (53), (58) will certainly be satisfied if

$$r\delta^\alpha(\rho) + |y(t, \rho)| \leqslant \epsilon(\rho) \qquad (0 \leqslant t \leqslant \delta(\rho)). \tag{59}$$

From the definition (57) of $y(t, \rho)$ it follows that

$$| y(t, \rho)| \leqslant \rho \,| X(t)|(1 + rt^{\alpha})^2 \sum_{k=0}^{\infty} \frac{\rho^k \,| X(t)|^k (1 + rt^{\alpha})^k}{(k + 2)!}$$
$$\leqslant \tfrac{1}{2}A\rho \,| X(t)|\, e^{A\rho|X(t)|}, \tag{60}$$

where

$$A = (1 + r\Delta^{\alpha})^2.$$

On account of Property 5. with $\alpha = \alpha'$ and (43) we can write

$$X(t) = t^{\alpha}\omega(t) \qquad (0 \leqslant t \leqslant \Delta) \tag{61}$$

with $\omega(0) = 0$, $\omega(t) \to 0$ if $t \downarrow 0$, which means, that because of the boundedness of $X(t)$ on $0 \leqslant t \leqslant \Delta$, there exists a function $\Omega(t)$, $\Omega(0) = 0$ and monotonically increasing, such that

$$| \omega(t)| \leqslant \Omega(t) \qquad (0 \leqslant t \leqslant \Delta). \tag{62}$$

Then it follows from (60), (61) and (62), that

$$| y(t, \rho)| \leqslant \tfrac{1}{2}B(\rho)\, e^{B(\rho)} \qquad (\rho \geqslant Q) \tag{63}$$

where

$$B(\rho) = A\rho\delta^{\alpha}(\rho)\Omega(\delta(\rho)). \tag{64}$$

From (59), (60) and (63) it appears that (55) is certainly satisfied if

$$r\delta^{\alpha}(\rho) + \tfrac{1}{2}B(\rho)\, e^{B(\rho)} \leqslant \epsilon(\rho) \qquad (\rho \geqslant Q). \tag{65}$$

In considering (56) and (65) $\delta(\rho)$ can be chosen in such a way that the relations

$$\rho\delta^{\alpha}(\rho) \to \infty \qquad (\rho \to \infty) \tag{66}$$

and

$$\rho\delta^{\alpha}(\rho)\Omega(\delta(\rho)) \to 0 \qquad (\rho \to \infty) \tag{67}$$

hold simultaneously. In fact, since $\Omega(u)$ is monotonically increasing on the interval $0 \leqslant t \leqslant \Delta$, and $\Omega(0) = 0$, the equation

$$\rho u^{\alpha} = \frac{1}{(\Omega(u))^{1/2}} \tag{68}$$

possesses for all sufficiently large ρ, say $\rho \geqslant \rho_1 \geqslant 1$, precisely one positive root $u = u(\rho)$ which is smaller than Δ. This root $u(\rho)$ is monotonically decreasing to zero if $\rho \to \infty$. We define

$$\delta(\rho) = u(\rho) \qquad (\rho \geqslant \rho_1) \tag{69}$$

and with this definition we put for all $\rho \geqslant \rho_2 \geqslant \rho_1$

$$\epsilon(\rho) = \max\{\zeta(\delta(\rho)), \delta^\alpha(\rho) + \tfrac{1}{2}B(\rho)\, e^{B(\rho)}\}, \tag{70}$$

$B(\rho)$ being defined in (64), where ρ_2 is chosen so large that for $\rho \geqslant \rho_2$ both quantities between the curled brackets are smaller than $\min(c, 1)$ and moreover, $B(\rho)$ is monotonically decreasing (to zero) for $\rho \geqslant \rho_2$. Then we define $Q = \rho_2$. With these definitions of $\delta(\rho)$, $\epsilon(\rho)$ and Q the assertion is proved and that means that for ν odd and $\rho \geqslant Q$ (54) and (55) are satisfied. Consequently, by (49), (50), (54), (55) and (48), $I_{\nu\rho}(\Delta)$ fulfils the following fundamental inequalities:

$$-\rho(1 + \tau\epsilon(\rho)) \int_0^{\delta(\rho)} t^\nu e^{-\rho(c+\epsilon(\rho))t^\alpha} X(t)\, dt \leqslant I_{\nu\rho}(\delta(\rho))$$
$$\leqslant -\rho(1 - \tau\epsilon(\rho)) \int_0^{\delta(\rho)} t^\nu e^{-\rho(c-\epsilon(\rho))t^\alpha} X(t)\, dt, \ (\nu \text{ odd}, \rho \geqslant Q). \tag{71}$$

Without knowing more about the behaviour of $X(t)$, i.e. of $\omega(t)$, if $t \downarrow 0$, it is impossible to derive from (71) much about the asymptotic behaviour of $I_{\nu\rho}(\delta(\rho))$ if $\rho \to \infty$. However, it is easy to show that

$$I_{\nu\rho}(\delta(\rho)) = o(\rho^{-a}) \qquad (\rho \to \infty) \tag{72}$$

with a given by (24). In fact, multiplying all three members of (71) with ρ^a and using (61), (62), it follows that

$$\rho^a \mid I_{\nu\rho}(\delta(\rho))| \leqslant \rho^{a+1}(1 + \epsilon(\rho))\{\delta(\rho)\}^{\nu+\alpha}\, \Omega(\delta(\rho)) \int_0^{\delta(\rho)} e^{-\rho(c-\epsilon(\rho))t^\alpha}\, dt.$$

Applying Lemma 6, this leads to

$$\rho^a \mid I_{\nu\rho}(\delta(\rho))| \leqslant CB(\rho)\, \delta^\nu(\rho)$$

where $B(\rho)$ is given in (64) and C is a properly chosen positive constant. Because of (64), (67) and the fact that $\delta(\rho)$ tends to zero if $\rho \to \infty$, (72) is true.

As a first corollary we show that from this result Theorem 5 can be proved again.

In fact, if ν is odd and $\rho \geqslant Q$, $I_{\nu\rho}(\delta)$ as given in (22) is written as

$$I_{\nu\rho}(\delta) = I_{\nu\rho}(\delta(\rho)) + \int_{\delta(\rho)\leqslant|t|\leqslant\delta} t^\nu \beta^\rho(t)\, dt. \tag{73}$$

Because of Property 5. with $\alpha = \alpha'$, $c = c'$, it is possible to choose $\delta_1 > 0$

and so small, that for all t with $|t| \leqslant \delta_1$, $\beta(t) > e^{-\frac{1}{2}ct^\alpha}$. Then $\delta(\rho) \leqslant \delta_1$ for sufficiently large ρ, say $\rho \geqslant \rho_3 \geqslant Q$ and

$$\rho^a \left| \int_{\delta(\rho) \leqslant |t| \leqslant \delta_1} t^\nu \beta^\rho(t)\, dt \right| \leqslant 2\rho^a \int_{\delta(\rho)}^{\delta_1} t^\nu e^{-\frac{1}{2}\rho c t^\alpha}\, dt$$
$$= 2\alpha^{-1}(2/c)^a \int_{2^{-1}c\rho\delta^\alpha(\rho)}^{2^{-1}c\rho\delta_1^\alpha} u^{a-1} e^{-u}\, du.$$

By (66) the latter integral tends to zero as $\rho \to \infty$.

Hence it follows from (73) and (72) that for odd values of ν

$$\lim_{\rho\to\infty} \rho^a I_{\nu\rho}(\delta_1) = \lim_{\rho\to\infty} \rho^a I_{\nu\rho}(\delta(\rho)) = 0.$$

Then, (41) holds and this again proves Theorem 5.

A second corollary to (71) will be treated in the next section.

5. Example to Property 5. of $\beta(t)$

In this section $\phi(t)$ and $\psi(t)$ in Property 5. of $\beta(t)$ are chosen in a special way. Because of (43) this means that in the fundamental relation (71) of $X(t)$ more is known and that will lead to a formula for the asymptotic behaviour of $I_{\nu\rho}(\delta(\rho))$.

It is assumed that Property 5. takes the following form, indicated by 5′:

5′ $\beta(t) = 1 - ct^\alpha + dt^\mu + \sigma(t)$ if $t \downarrow 0$,

with $\mu > \alpha > 0$, $c > 0$, $d \neq 0$, $\sigma(t) = o(t^\mu)$,

$\beta(t) = 1 - c\,|t|^\alpha + d'\,|t|^{\mu'} + \tau(t)$ if $t \uparrow 0$,

with $\mu' > \alpha > 0$, $d' \neq 0$, $\tau(t) = o(|t|^{\mu'})$.

In the investigation it will be supposed that $\mu < \mu'$ because its conclusions appear to hold with only minor changes if $\mu' < \mu$ or $\mu = \mu'$.

Then in (43),

$$X(t) = -dt^\mu + d't^{\mu'} - \sigma(t) + \tau(-t) = t^\alpha\omega(t), \tag{74}$$

where, in accordance with (61),

$$\omega(t) = -dt^{\mu-\alpha} + a(t), \tag{75}$$

with

$$a(t) = o(t^{\mu-\alpha}) \qquad (t \downarrow 0). \tag{76}$$

In (62) we may choose

$$\Omega(t) = Dt^{\mu-\alpha} \qquad (0 \leqslant t \leqslant \Delta)$$

where $\Delta > 0$ is so small that on the interval $0 < t \leqslant \Delta$ $X(t) \neq 0$ and $X(t)$ has a fixed sign; $D > 0$ is properly chosen. On $0 < t \leqslant \Delta$ is

$$\tau = sgn(-d) \tag{77}$$

because of (45). (66) and (67) take the form

$$\rho\delta^{\alpha}(\rho) \to \infty \qquad \text{and} \qquad \rho\delta^{\mu}(\rho) \to 0 \qquad (\rho \to \infty)$$

respectively, from which it follows that in the special case, considered in this section, we may take

$$\delta(\rho) = \rho^{-\gamma}\left(\frac{1}{\mu} < \gamma < \frac{1}{\alpha}\right).$$

Because of the results of Sub-section 4.4, with this choice of $\delta(\rho)$ it is possible to construct for all sufficiently large ρ ($\rho \geqslant Q$) a positive function $\epsilon(\rho)$ with $\epsilon(Q) < \min(c, 1)$ and monotonically decreasing to zero if $\rho \to \infty$, such that for all $\rho \geqslant Q$ the fundamental inequalities (71) hold, i.e.

$$-\rho(1 + \tau\epsilon(\rho)) \int_0^{\rho^{-\gamma}} t^{\nu} e^{-\rho(c+\epsilon(\rho))t^{\alpha}} X(t)\, dt \leqslant I_{\nu\rho}(\rho^{-\gamma})$$
$$\leqslant -\rho(1 - \tau\epsilon(\rho)) \int_0^{\rho^{-\gamma}} t^{\nu} e^{-\rho(c-\epsilon(\rho))t^{\alpha}} X(t)\, dt,$$

from which, with $X(t)$ written in the form

$$X(t) = t^{\mu}(-d + b(t)), \qquad \text{and} \qquad b(t) = t^{\alpha}a(t) = o(1) \qquad \text{if} \quad t \downarrow 0, \tag{78}$$

(because of (74), (75), (76)), it follows that

$$\begin{aligned} -\rho(1 + \tau\epsilon(\rho)) \int_0^{\rho^{-\gamma}} t^{\mu+\nu} e^{-\rho(c+\epsilon(\rho))t^{\alpha}}(-d + b(t))\, dt \leqslant I_{\nu\rho}(\rho^{-\gamma}) \\ \leqslant -\rho(1 - \tau\epsilon(\rho)) \int_0^{\rho^{-\gamma}} t^{\mu+\nu} e^{-\rho(c-\epsilon(\rho))t^{\alpha}}(-d + b(t))\, dt. \end{aligned} \tag{79}$$

Applying Lemma 6,

$$\int_0^{\rho^{-\gamma}} t^{\mu+\nu} e^{-\rho(c+j\epsilon(\rho))t^{\alpha}}\, dt = \frac{1}{\alpha\{\rho(c + j\epsilon(\rho))\}^{g}} \{\Gamma(g) - W_i(\rho)\}, \tag{80}$$

where $i = 0, 1; j = (-1)^i$, $g = (\mu + \nu + 1)/\alpha$ and

$$W_i(\rho) = \int_{(c+j\epsilon(\rho))\rho^{1-\alpha\gamma}}^{\infty} u^{g-1}e^{-u}\, du. \tag{81}$$

Further, with respect to $b(t)$, defined in (78) and combined with (76) it is clear that to each ϵ_b with $0 < \epsilon_b \leqslant \frac{1}{2}$ there exists a $Q_b \geqslant Q$ such that for all $\rho \geqslant Q_b$

$$|\, b(t)| \leqslant -\tau\, d\epsilon_b \qquad (0 \leqslant t \leqslant \rho^{-\gamma}). \tag{82}$$

Combining (79), (80) and (82) the following inequalities are obtained

$$\frac{(1 + \tau\epsilon(\rho))(1 + \tau\epsilon_b)\, d}{\alpha(c + \epsilon(\rho))^g} \{\Gamma(g) - W_0(\rho)\} \leqslant \rho^{g-1} I_{\nu\rho}(\rho^{-\gamma}) \tag{83}$$

$$\leqslant \frac{(1 - \tau\epsilon(\rho))(1 - \tau\epsilon_b)\, d}{\alpha(c - \epsilon(\rho))^g} \{\Gamma(g) - W_1(\rho)\}.$$

Because of the fact that $1 - \alpha\gamma > 0$, it follows from (81) that if $\rho \to \infty$

$$\rho^{g-1} W_i(\rho) \to 0 \quad (i = 0, 1),$$

and, using an argument as in Sub-section 4.2 (83) results in

$$\lim_{\rho\to\infty} \rho^{g-1} I_{\nu\rho}(\rho^{-\gamma}) = d\alpha^{-1}c^{-g}\Gamma(g)\, (\nu \text{ odd}).$$

From this result it follows by reasoning as in the first corollary in Sub-section 4.4 that

$$I_{\nu\rho}(\delta) = d\alpha^{-1}c^{-g}\Gamma(g)\, \rho^{1-g} + o(\rho^{1-g}) \qquad (\nu \text{ odd}, \rho \to \infty).$$

Taking $\nu = 1$ and using (46) with $\nu = 0$, combined with (5) and Lemma 3, this results in

$$\frac{I_{1\rho}(\delta)}{I_\rho} = d2^{-1}c^{-(\mu+1)/\alpha}\Gamma((\mu + 2)/\alpha)\{\Gamma(1/\alpha)\}^{-1}\, \rho^{-(\mu+1-\alpha)/\alpha} + o(\rho^{-(\mu+1-\alpha)/\alpha}). \tag{84}$$

Consequently, by (18), (84) and (47) together with Lemma 1, the following formula holds for $\rho \to \infty$:

$$\begin{aligned} U_\rho(f; x) - f(x) = &-d2^{-1}c^{-(\mu+1)/\alpha}\Gamma((\mu + 2)/\alpha)\{\Gamma(1/\alpha)\}^{-1} f'(x)\, \rho^{-(\mu+1-\alpha)/\alpha} \\ &+ 2^{-1}c^{-2/\alpha}\Gamma(3/\alpha)\{\Gamma(1/\alpha)\}^{-1} f''(x)\, \rho^{-2/\alpha} \\ &+ o(\rho^{-(\mu+1-\alpha)/\alpha}) + o(\rho^{-2/\alpha}). \end{aligned}$$

This result leads to

THEOREM 7. *If $\beta(t) \in B$ and $\beta(t)$ possesses Property 5'. with $\mu < \mu'$ and $d \neq 0$, if $f(t) \in M$ and $f''(x)$ exists at a point $t = x$, then*

$$\rho^{\sigma/\alpha}\{U_\rho(f; x) - f(x)\} = p(x) + o(1) \qquad (\rho \to \infty),$$

where

$$\sigma = \min(\mu + 1 - \alpha, 2)$$

and

(i) if $0 < \mu - \alpha < 1$, then $\sigma = \mu + 1 - \alpha$ and

$$p(x) = -d2^{-1}c^{-(\mu+1)/\alpha}\Gamma((\mu + 2)/\alpha)\{\Gamma(1/\alpha)\}^{-1}f'(x),$$

(ii) if $\mu - \alpha = 1$, then $\sigma = 2$ and

$$p(x) = 2^{-1}c^{-2/\alpha}\Gamma(3/\alpha)\{\Gamma(1/\alpha)\}^{-1}\{-3d(\alpha c)^{-1}f'(x) + f''(x)\}$$

(iii) if $\mu - \alpha > 1$, then $\sigma = 2$ and

$$p(x) = 2^{-1}c^{-2/\alpha}\Gamma(3/\alpha)\{\Gamma(1/\alpha)\}^{-1}f''(x).$$

ADDENDUM. *If on the contrary $\mu' < \mu$ and $d' \neq 0$, then in the assertions of theorem 7, μ is to be replaced by μ' and in* (i), (ii), (iii) *d by $-d'$.*
If $\mu = \mu'$, and $d \neq d'$, in (i), (ii) *d is to be replaced by $d - d'$.*

6. APPLICATIONS

In this section we consider a special case of Theorem 7, which was proved in a study [7], preceding the present paper.

THEOREM 8. *If $\beta(t) \in B$ and $\beta'''(0)$ exists, while $\beta''(0) \neq 0$, if $f \in M$ and $f''(x)$ exists at a point $t = x$, then*

$$\rho\{U_\rho(f; x) - f(x)\} = \frac{-1}{2\beta''(0)}\left\{\frac{\beta'''(0)}{\beta''(0)}f'(x) + f''(x)\right\} + o(1)\ (\rho \to \infty). \qquad (85)$$

Proof. As $\beta'''(0)$ exists, $\beta(t)$ can be written as

$$\beta(t) = \beta(0) + t\beta'(0) + \tfrac{1}{2}t^2\beta''(0) + \tfrac{1}{6}t^3\beta'''(0) + t^3\kappa(t),$$

in which, because of the fact that $\beta \in B$, $\beta(0) = 1$, $\beta'(0) = 0$, $\beta''(0) < 0$, Hence, $\beta(t)$ possesses property 5'. if $\beta'''(0) \neq 0$, with $\alpha = 2$, $\mu = \mu' = 3$, $c = -\tfrac{1}{2}\beta''(0)$, $d = \tfrac{1}{6}\beta'''(0)$, $d' = -\tfrac{1}{6}\beta'''(0)$, $t^3\kappa(t) = o(t^3)$ if $t \to 0$. Thus, if $\beta'''(0) \neq 0$, Theorem 7, together with its addendum results in $\sigma = 2$ and (ii) then gives (85).

Of the well-known operators several are of the type considered in Theorem 8. As examples we mention here

1. $\beta(t) = e^{-t^2}$, $I_\rho = (\pi/\rho)^{1/2}(\rho \geqslant 1)$ (Weierstrass [10]).

(85) takes the form

$$\rho \left\{(\rho/\pi)^{1/2} \int_{-\infty}^{\infty} f(x-t)\, e^{-\rho t^2}\, dt - f(x)\right\} = \tfrac{1}{4}f''(x) + o(1)\ (\rho \to \infty).$$

2. $\beta(t) = 1 - t^2(|\, t\,| \leqslant 1)$, $\beta(t) \equiv 0\ (|\, t\,| > 1)$ (Landau [5]).

Then, if $\rho \geqslant 1$,

$$I_\rho = B(1/2, \rho + 1) = (\pi/\rho)^{1/2}(1 + o(1)) \qquad (\rho \to \infty).$$

Theorem 8 gives

$$\rho \left\{(\rho/\pi)^{1/2} \int_{-1}^{1} f(x-t)(1-t^2)^\rho\, dt - f(x)\right\} = \tfrac{1}{4}f''(x) + o(1)\ (\rho \to \infty).$$

3. $\beta(t) = \cos^2(\pi/2)t(|\, t\,| \leqslant 1)$, $\beta(t) \equiv 0\ (|\, t\,| > 1)$.

Then, if $\rho \geqslant 1$, $I_\rho = (8/(\pi\rho))^{-1/2}$.
The corresponding operators are the slightly modified de la Vallée-Poussin operators [9]. Theorem 8 gives

$$\rho \left\{(\pi\rho/8)^{1/2} \int_{-1}^{1} f(x-t)(\cos(\pi/2)\, t)^{2\rho}\, dt - f(x)\right\} = (4/\pi^2)\, f''(x) + o(1)\ (\rho \to \infty).$$

It is to be noticed that in all three above examples $\beta(t)$ is an even function. An example where this is not so is the following one:

4. $\beta(t) = e^{-t^2+t^3}\ (|\, t\,| \leqslant \tfrac{1}{2})$, $\beta(t) \in B$.

Then, if $\rho \geqslant 1$, $I_\rho = (\pi/\rho)^{1/2}(1 + o(1))$. Theorem 8 is applicable with $\alpha = 2$, $\mu = \mu' = 3$, $\beta''(0) = -2$, $\beta'''(0) = 6$ and it gives

$$\rho \left\{(\rho/\pi)^{1/2} \int_{-\infty}^{\infty} f(x-t)\, \beta^\rho(t)\, dt - f(x)\right\} = -\tfrac{3}{4}f'(x) + f''(x) + o(1)\ (\rho \to \infty).$$

A first example where Theorem 8 is not applicable is a generalisation of the above Example 2:

5. $\beta(t) = 1 - |\, t\,|^\alpha(\alpha > 0, |\, t\,| \leqslant 1)$, $\beta(t) \equiv 0\ (|\, t\,| > 1)$.

Then, if $\rho \geqslant 1$,

$$I_\rho = 2\alpha^{-1}B(\alpha^{-1}, \rho + 1) = 2\alpha^{-1}\Gamma(1/\alpha)\, \rho^{-1/\alpha}(1 + o(1)) \qquad (\rho \to \infty),$$

$B(x, y)$ denoting Euler's beta function. According to Theorem 7 with $c = 1$, we have

$$\rho^{2/\alpha}\left\{\rho^{1/\alpha}\{2\Gamma((1/\alpha)+1)\}^{-1}\int_{-1}^{1} f(x-t)(1-|t|^{\alpha})^{\rho}\,dt - f(x)\right\}$$
$$= 2^{-1}\Gamma(3/\alpha)\{\Gamma(1/\alpha)\}^{-1} f''(x) + o(1)\ (\rho \to \infty).$$

The case $\alpha = 2k$ ($k \in \mathbf{N}$, $k \geqslant 2$) is due to Mamedov [6].
A second one is the following one:

6. $\beta(t) = e^{-t^4+t^5}$ ($|t| \leqslant \frac{1}{2}$), $\beta(t) \in B$.

Then, if $\rho \geqslant 1$,

$$I_\rho = 2^{-1}\rho^{-1/4}\Gamma(1/4)(1+o(1)) \qquad (\rho \to \infty).$$

Application of Theorem 7, together with its addendum, gives

$$\rho^{1/2}\left\{2\rho^{1/4}\{\Gamma(\tfrac{1}{4})\}^{-1}\int_{-\infty}^{\infty} f(x-t)\,\beta^{\rho}(t)\,dt - f(x)\right\}$$
$$= 2^{-1}\Gamma(\tfrac{3}{4})\{\Gamma(\tfrac{1}{4})\}^{-1}\{-1\tfrac{1}{2}f'(x) + f''(x)\} + o(1)\ (\rho \to \infty).$$

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